Some Properties Z-S Semi Models

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Abstract. Let H unital (left) E–module where E commutative ring with identity. As generalization of small prime submodule we present and discuss the idea of Z-small prime submodule. Among the result that we obtain the following: a submodule W of a finitely faithful multiplication E – module. If W is Z-S-P submodule of H, then [W:H] is a Z-S-P- ideal of E.

Introduction

Let W be a submodule and not equal of an E-module H then W is called prime if whenever e ∈ E , h∈ H and eh ∈ W gives us either h ∈ H or e ∈ [W:H], [1-3]. As a generalization of this concept [4] suggest idea of small prime submodule , where a proper submodule W of an E-module H is small prime if e ∈ E , x ∈ H , < x > is small in H and ex ∈ W , then either x ∈ W or e ∈ [W:H] . A submodule W of an R-module is called small (denoted by W≪ H) if for every N ⊆ H with W+N = H implies N = H .

And W ⊂ H (H is E-module) is tremed a Z-small submodule (in symble W ≪z H) if W+N = H, where N ⊆ H with N ⊇ Z2/(H) , then N = H , where Z2/(H) is defined by Z (H/Z(H)) =Z2(H)/Z(H). We introduce the concept of Z-small prime submodule(in short Z-S-P) , where submodule W of E-module (W ⊂ H)is called Z-S-P submodule (denoted by W≪Z−P H) if and only if whenever e ∈ E , x ∈ H , with < x > ≪z H and ex ∈ W gives us either e ∈ [W:H] or x ∈ W.

This paper consist of three sections , in section one we give an introduction for this paper . In section two we present the idea of Z-small prime submodule and study many properties of this concept. In section three we study the relation of this concept and another

Mothedlogy of research

Concept of submodules. Z–S-P Submodules In this section we study the concept of Z–S-P submodule, and we provide some examples and basic properties about it.

Definition 2 .1:

A proper submodule W of H (H is E − module)is called a Z–small prime submodule and it is denoted by (W ≪Z−P H) if and only if whenever e ∈ E , x ∈ H , and < x > ≪z H and rx ∈ W we have either e ∈ [W:H] or x ∈W.

Actually ideal L of ring E is termed Z-S-P if L is a Z-S-P submodule of E-module E . Similarly, actually ideal K of a ring E is Z-small prime ideal if and only if whenever s ,e ∈ E with < s > ≪zE and es ∈ K we have either e ∈ L or s ∈ K .

Remark and examples 2 .2:

1. Let W≪Z−P H then W be a small prime submodule.

Proof:

Let 0 ≠ W ⊂ H, and H is an E-module which is Z-small prime.

Let $e \in E$, $x \in H$ with $< x > \ll H$ and $ex \in W$. Since every small submodule is Z-small by [5] , so $< x > \ll_z H$. But W is Z-S-P submodule so either $e \in [W:H]$ or $x \in W$.
The reverse is not correct largely : For that , consider H = Z24 as a Z-module and $W = < \bar{6} > = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\}$. W is small prime submodule by [4] but not Z-S-P submodule. To prove that W is not Z-S-P submodule of H , we have [W:H]=6Z. Notice that $\bar{0} = 8 \cdot \bar{3}$, and $< \bar{3} > \ll_Z H$, but $\bar{3} \notin W$ and $8 \notin [W:H]$, thus W is not Z–small
prime submodule.
2. Let H be an E-module and <x> is small(xεH) then every small prime submodule is Z-S-P . (This is the converse of (1) under certain condition)
Proof :
(⟹) From (1)
(⟸) Let $0 \neq W \subset H$ such that ex □ W , e □ E , x □ H with $< x > \ll_Z H$. But by assumption $< x > \ll H$ and again by assumption either x □ W or e □ [W:H] .
3. Let W be prime submodule of E-module H then each prime submodule is Z-S-P .
4. The reverse of (3) cannot be true in general : take an example W=8Z , H=Z as a Z-module we have $W \ll_{Z-P} H$, but it's not prime since $2 \cdot 4 \in 8Z$, $4 \notin 8Z$ and $2 \notin [8Z: Z] = 8Z$.
5. Every proper submodule W of a hollow E-module H is Z-S-P submodule if and only if W is small prime submodule "where E-module $H \neq 0$ is termed Z-hollow if every $W \subset H$ is Z-small" [5].
Proof :
(⟹) From (1)
(⟸) Let ex □ W , e □ E , x □ H with $< x > \ll_Z H$. By assumption $< x > \ll H$. So by assumption we get the result
6. Let $W \ll_{Z-P} H$ and let F □ W then F may not Z-S-P submodule , for example:
let $W = < \bar{2} >$ is a Z-S-P of Z24 but $F = < \bar{8} > \subseteq W$ is not Z-S-P since $2 \cdot \overline{12} \in < \bar{8} >$ but $\overline{12} \notin < \bar{8} >$ and $2 \notin [< \bar{8} >:_z W] = 8Z$.
7. A direct summand of Z-S-P submodule in general cannot be Z-S-P submodule .
Example : consider H = Z24 as a Z-module and $W = < \bar{2} >$, it is clear that $W = < \bar{6} > \oplus < \bar{8} >$, while $< \bar{6} >$ is not $\ll_{Z-P} H$ since $8 \cdot \bar{3} \in < \bar{6} >$ but $\bar{3} \notin < \bar{6} >$ and $8 \notin [< \bar{6} >:_z W] = 3Z$.
8. Every nontrivial submodule of Z-module Z6 is Z-s-P , but $< \bar{0} >$ is not a Z-S-P since $2 \cdot \bar{3} \in < \bar{0} >$ with $\bar{3} \notin < \bar{0} >$ and $2 \notin [< \bar{0} >:_z Z6] = 6Z$.
9. If D<W<H , and H is an E-module with D is a Z-S-P in W , W is Z-small in H , this is not necessary lead to $D \ll_{Z-P} H$, by the following example shown:
Let $D = < \bar{4} >$, $W = < \bar{2} >$ be submodules of Z24. As we see that W is Z-small prime submodule of Z24 and one can easily show that D is Z-S-P in W , yet D is not Z-S-P in Z24 since $\bar{4} = 2 \cdot \bar{2} \in < \bar{4} >$, and $< \bar{4} > \ll_Z Z24$. But $\bar{2} \notin < \bar{4} >$ and $2 \notin [D: Z24] = 4Z$.
10. Let $0 \neq G$ be an ideal of the ring Z such that $G \neq Z$, then G is a Z-S-P ideal , since Z is an integral domain with < 0 > is the only Z-small ideal of Z .
11. If $L \subseteq H \subseteq F$ with $H \ll_{Z-P} F$, then it is not necessarily $L \ll_{Z-P} H$ for example: let $L = < \overline{12} >$, $H = < \bar{2} >$ in the Z-module Z24 as we show that H is Z-small prime submodule in Z24 . But L is not Z-S-P in H since $3 \cdot \bar{8} \in L$, and $< \bar{8} > \ll_Z Z24$. But $\bar{8} \notin L$ and $3 \notin [L:H] = 6Z$.
12. Every Z–small submodule is a Z-S-P submodule, but the converse cannot be true in general, this is because in the Z −module Z , if we take A = 3Z , then A is a Z-S-P

submodule , yet it is not Z–S submodule .13. If W≪Z−P H then it is not necessary be annEW prime ideal of E .Such as: if H = Z24 a Z-module and W = < $\bar{2}$ > , thus ann ⟨$\bar{2}$⟩ = 12Z is not prime ideal of Z since 3 · 4 = 12 ∈ 12Z , but 3 and 4 ∉ 12Z . 14. If H and W are submodules of an E-module F and H+W is Z −small prime in Z , but in general , it is not necessarily that H and W ≪Z−P Z for example:let H = < $\bar{4}$ > , W = < $\bar{6}$ > are submodules of Z-module Z12 .
Notice that , H+W = < $\bar{2}$ >. One can easily show that < $\bar{2}$ > ≪Z−P, but H is not Z-small prime submodule of Z12 since 2 · $\bar{2}$ ∈< $\bar{4}$ > , but $\bar{2}$ ∉ < $\bar{4}$ > , < $\bar{2}$ >≪z Z12 and 2 ∉
[H: Z12] = 4Z also W is not Z-small prime submodule in Z12 since 2 · $\bar{3}$ ∈ < $\bar{6}$ > , <$\bar{3}$ > ≪z Z12 but $\bar{3}$ ∉ < $\bar{6}$ > and 2 ∉ [W: Z12] = 6Z .
15. Consider F as E-module and L ideal of E where L ⊆ ann(F). Let W ⊆ F then W is Z-S-P submodule of F if and only if W is Z-S-P E/L of F .
Proof: Let r + l ∈E/L, x ∈ F such that < x >≪z F and (r + l)x ∈ W . As L ⊆ an F ,
we have (r + L)x = rx ∈ W , but W is a Z-small prime submodule of F , so either r ∈
[W:F] or x ∈ E . Hence, the result follows easily .

Proposition 2 .3: [6]

Let T be a maximal submodule of an E-module F then [T:F] is maximal ideal of E .

Proposition 2 .4: [7]

Let T be a maximal submodule of an E-module F , if [T: F] is a maximal ideal , then T will be a prime submodule .

Corollary 2 .5:

Let T ⊆ F , if we have [T : F] as a maximal ideal , then T ≪Z−P F .

Proof:

Clear by Propositions 2 .2 and 2 .4, respectively.

Corollary 2 .6:

If W is a maximal of E-module H, then W is prime (W ⊂ H)

Proof:

Clear by Propositions 2 .4, 2 .1 and Corollary 2 .5.

Corollary 2 .7: [6]

Let H be an E-module, consider I as maximal ideal of E . If IH ≠ H, so IH is aprime submodule of H.“ Recall E-module H is named multiplication, if every W ⊆ H, W = [W:H]H where[W:H] = {e ∈ E : eH ⊆ W}” [8]“Recall E-module H is faithful if its annihilator is zero” [9].

Theorem 2 .8:

Consider H is a finitely generated faithful multiplication E - module and W ⊂ H, so W≪Z−P H if and only if [W:H] is a Z-prime ideal of E.

Proof:

Let r , s ∈ E and < s > ≪Z E with rs ∈ [W:H] then (rs)H ⊆ W , but < rs > ⊆ < s > ≪Z E , then < rs > ≪Z E , by [5] . So, H is finitely generated faithful multiplication E- module, thus (rs)H ≪Z H, by [5], therefore either s ∈ [W:H] or r ∈ [W: H] , and so [W:H]is a Z-small prime ideal of E.

Proposition 2 .9:

Consider W and L are Z-small prime submodules of H , and H is an E - module such that [W:H] = [L:H] . So, W∩L ≪Z−P C .

Proof:

Let e ∈ E ,x ∈ H with < x > ≪z H and ex ∈ W∩L , then x ∈ W and ex ∈ L . Since W , L are Z–small prime submodule then either e∈[W:H] or x ∈ W and either e∈[L:H] or x ∈ L . Hence, x ∈ W and x ∈ L or e ∈ [W:H] = [L:H]. Which implies x ∈ W∩L or e ∈ [W∩L :H] . Therefore, ∩L ≪Z−P H .

Proposition 2 .10:
If W ≪Z−P H and H is an E −module with I will be any ideal of E, then we will have [W:I] ≪Z−P H .
Proof:
Let e ∈ E , x ∈ H with < x > ≪z H and x∈[W : I] .Then exI ≤ W . Such that exa ∈ W ∀ a ∈ I . But < xa > ≤ < x > . Then < xa > ≪z H by [5] therefore , either xa ∈ W ∀ a ∈ I or e ∈[W:H] ≤ [[W:I]:H] . So , either xI ≤ W or e ∈ [[W:I]:H] . Thus either x ∈ [W:I] or e ∈ [[W:I]:H] and that is what we want to prove .
Remark 2 .11:
The converse of previous proposition isn't correct largely.By way of illustration: H = Z45 a Z module and W = < $\overline{15}$ > , I = 3Z then[W:H I] = [< $\overline{15}$ >:H 3Z] = < $\overline{5}$ > is Z–S-P submodule of H But W is not Z–S-P submodule of H because 6 · 5 ∈ < $\overline{15}$ > , < $\overline{5}$ > ≪Z Z45 , but $\overline{5}$ ∉ < $\overline{15}$ > and 6 ∉ [< $\overline{15}$ >: Z45] = 15Z .
Remark 2 .12:
If L < W < H and W is Z–S-P submodule of H so it is not necessarily L≪Z−P H By way of illustration : let H = Z24 as a Z- module and let L = < $\overline{6}$ > , W = < $\overline{2}$ > and [L:H] = 6Z , [W:H]=2Z. Then W ≪Z−P H but L isn't Z-small prime submodule of H . because 8 · $\overline{3}$ ∈ L with < $\overline{3}$ >≪Z H but $\overline{3}$ ∉ L and 8 ∉ [L:H]=6Z .
Remark 2 .13:
If L < W < H and L is Z–S-P submodule of W so it is not necessarily W≪Z−P H . By way of illustration: let H = Z24 as a Z -module and let L = < $\overline{12}$ > , W= < $\overline{6}$ > and [L:W] = 2Z , [W:H] = 6Z .Then L is Z–S-P submodule of W but W is not Z–S-P submodule of H . Because 8 · $\overline{3}$ ∈L with < $\overline{3}$ >≪Z H but $\overline{3}$ ∉ L and 8 ∉ [L:H]=6Z .
Remark 2 .14:
If L ,W are submodule of a module H such that L≪Z−P W then it is not necessarily L is Z–S-P submodule of H . If we take an example: let H = Z24 as aZ- module and let L = < $\overline{8}$ > , W = < $\overline{4}$ > and [L:H] = 8Z . Then L is Z–S-P submodule of W. But L is not Z–S-P submodule of W because $\overline{0}$ = 2 · $\overline{12}$ with < $\overline{12}$ >≪ZH but $\overline{12}$ ∉ L and 2 ∉ 8Z.
Another result about Z–S-P submodules
In this section many properties of Z–S-P submodule are given. Moreover, we study the inverse image of Z-small prime sub module.
Result of research
Proposition 3 .1:
Let f: H → H̀ be E- epimorphism and assume L ≪Z−P H̀ . So, we will have f−1(L)≪Z−P H .
Proof:
Let ra ∈ f−1(L) where e ∈ E , a ∈ H , < a >≪z H , then f(ea) = ef(a) ∈L . Since < a > ≪z H then < f(a) > ≪z H̀ by ([5, Proposition 2 .3) And since L is Z–small prime submodule in H̀ implies either f(a) ∈ L or e ∈ [L: H] If f(a) ∈ L then a ∈ f−1(L) If e ∈ [L: H], then eH̀ ≤L and hence ef(H) = f(eH) ≤ L which gives eH̀ ≤ f−1(L). That means e ∈ [f−1(L): H]. So, f−1(L) ≪Z−P H .
Remark 3 .2:
Let L ≪Z−P H then a homomorphic image of L is not largely Z–S-P submodule. By way of illustration: f: Z45 → Z45 defined by h($\overline{x}$) = 3$\overline{x}$, ∀ $\overline{x}$∈ Z45 notice that the submodule < $\overline{5}$ > ≪Z−P Z45, but h(< $\overline{5}$ >) = < $\overline{15}$ > is not Z-small prime submodule of Z45 as we proved in Remark 2 .11 .
Corollary 3 .3:
Take L ,K submodules of E- module H with K ≤ L and L/K≪Z−P H/K then L ≪Z−P H .

Proof:
Let π: H →H/K be a natural epimorphism, and since L/K is Z-S-P submodule of H then by K π^{-1}(L/K) is Z − S − P submodule of H , so L will be Z-S-P submodule ofH .

Remark 3 .4:
The converse of previous corollary is not hold in general Take an example: consider H = Z24 be a Z-module and L = < $\bar{2}$ > , K = < $\bar{8}$ > , and K≤ L . L is Z-S-P submodule of Z24 but L/K ≅ < $\bar{6}$ > is not Z-S-P submodule of H/K≅< $\bar{3}$ > .

Corollary 3 .5:
Let H be an E-module and A ≤ B ≤ C ≤ H with CB is Z-S-P submodule of H B then C/A ≪Z−P H/A.

Proof:
Let h:H/A→H/B be the map defined by h(x + A) = x + B , ∀x ∈ E . Clearly h is an epimorphism. Since C/B is Z − S − P submodule of H/B, then by Proposition 3 .1 , f−1 (C/B) is Z– small prime submodule of H/A that means C/A Z − S − P submodule of H/A

Proposition 3 .6:
Let C be finitely generated faithful multiplication E-module . If A is Z-S-P submodule of C so [A:C] is Z-S-P ideal of E .

Proof:
Take x ,a ∈ E with < a >≪z E and xa ∈ [A:C] . then (xa)C ≤ A . But < xa > ≤ < a > ≪z E implies that < xa > ≪z E by [5].And since C is f . g faithful multiplication E-module, therefore (xa)C≪z C .Hence either e ∈ [A:C] or a ∈ [A:C] and hence [A:C] is Z-S-P ideal of E. However, we present the next proposition :

Proposition 3.7:
Let M and L be two submodules of an E-module H such that L ≪Z−P H and M is not contained in L. Then L ∩ M ≪Z−P M.

Proof:
Since M ≰ L ,so L ∩ M will be a proper submodule of M .Now, let e ∈ E , m ∈ M and em ∈ L ∩ M and < m > ≪Z M . Suppose m ∉ L and since < m > ≪Z H by [6 , Th . 2 .2] . and L ≪Z−P H thus r ∈ [L:H]. Thus eH ⊆ L and hence em ⊆ L ∩ M .that is e ∈ [L ∩ M:M] and this means that L ∩ M is Z-small prime of M.

Proposition 3 .8:
Let H1 and H2 be two E-module and let = H1 ⨁ H2 . If L = L1 ⨁ L2 is Z-S-P submodule of H, so L1 & L2 are Z-S-P submodules of H1 & H2 , subsequently .

Proof :
To prove L1 ≪Z−P H1 , let r ∈ R and h1 ∈ H1 and rh1 ∈ L1 with < h1 > ≪Z L1 thus r(h1 ,0) ∈ L1 ⨁ L2. By [5] if H = H1 ⨁ H2 , L = L1⨁L2 , where L1 ≤ H1 , L2 ≤ H2 then L ≪Z H if L1 ≪Z H1 and L2 ≪Z H2 we have < h1 ,0 >≪Z H1⨁H2 , but L1 ⨁ L2 is small prime of H , so either (h1 ,0) ∈ L1 ⨁ L2 or r ∈ [L1 ⨁ L2: H1 ⨁ H2] . Then either h1 ∈ L1 or r ∈ [L1: H1] ∩ [L2: H2] and hence either h1 ∈ L1 or r ∈ [L1: H1] this mean that L1 is z-small prime submodule of H. In the similar way one can easily show that L2 is Z-small prime of H2 .

Definition 3 .9: [7]
Let H be an E-module an E-submodule L of H is termed primary submodule if L ≠ N and wherever rx ∈ L , r ∈ E and x ∈ H we have either x ∈ L or r n∈[L:H] , for some n ∈ Z where [L:E H] = {r :

r ∈ E and Rh ⊆ L} .

Remark 3 .10:

There is no relation between Z-small prime submodules and primary submodules as the next examples show:

Take < $\bar{4}$ > of Z-module Z24 is not Z-S-P but it is primary .The submodule < $\overline{15}$ > in Z-module Z45 is Z-small prime, but it is not primary since 3 . $\bar{5}$ ∈< 15 > but $\bar{5}$ ∉ < $\overline{15}$ > and 3n ∉ [< $\overline{15}$ >:Z Z45] = < $\overline{15}$ > , for all n ∈ Z+ The following proposition gives the relation between Z-small prime submodule and primary submodule under certain condition Recall that, “a proper ideal I of a ring E is called semi-prime ideal if whenever x□E such that x∈ I then x ∈ I.

Proposition 3 .11: Let W be a primary submodule of an E-module H and [W :E H] is semi prime ideal of E, then W will be Z-small submodule of H.

Proof:

Assume W is a primary submodule with [W:E H] semi prime ideal , let rm ∈ W , r ∈ E, h ∈ H and suppose h ∉ W but W is a primary submodule, so r^n∈ [W: E H] , for some n ∈ Z and hence r ∈ √[W:E H] , but [W:E H] is semi prime , hence r∈ [W:E H] thus W is Z-small prime submodule of H .

Definition 3 .12: [11]

Let H be an E-module , then Z(H) = {x ∈ H , ann(x) ≤e E} is called the singular submodule of H.

Proposition 3 .13:

Let H be an E-module such that every cyclic submodule is singular. Then every Z-S-P submodule is prime submodule.

Proof:

Let W be a Z-S-P submodule of H and let ra ∈ W, if < a > ⊆ H then < a > is singular so by [5], < a > ≪z H. But W Z-S-P submodule so either a ∈ W or r ∈ [W:H] so W is prime submodule.

Conclusion

In this paper we succeed to present the concept of Z-small prime submodule and we showed that it is an expand to the concept of small prime submodule . And we presented some important relations between Z-small prime submodules and other kinds of modules.

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